# Current algebra and soft pionic modes in asymmetric quark matter

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**Abstract**

Pionic excitations are analyzed in isospin-asymmetric Fermi gas of constituent quarks. It is found that in addition to the usual pionic excitations, there exists a very low collective mode with quantum numbers of the charged pion (spin-isospin sound). In the chiral limit the excitation energy of one of the modes scales as the current quark mass, and the mode saturates current-algebraic sum rules. This is in agreement with predictions of current algebra for isospin-asymmetric medium.

Recently chiral current algebra has been applied [1,2] to systems with finite chemical potentials in order to study excitations in such systems. A formal prediction made by Cohen and one of us (WB) is that in the isospin-asymmetric nuclear medium there exists an excitation with quantum numbers of the charged pion ($\pi^+$ for medium with negative isospin density), which in the chiral limit becomes very soft [1]. Its excitation energy is proportional to the current quark mass $m$, and not $\sqrt{m}$ as in the case of the vacuum, and in this sense the mode is softer than the usual pseudo-Goldstone boson.[1] If such a soft mode exists at the physical value of $m$ in dense nuclear medium, and if it is strongly coupled, it would bring important phenomenological consequences by dominating excitation functions and thermal properties of the system at low energies. Unfortunately, current algebra gives us no hint as to whether

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[1] Note that in isospin-asymmetric medium the Goldstone theorem does not apply for charged excitations, since the constraint breaks explicitly the appropriate symmetry [1].

the physical value of $m$ is low enough such that the formal results of Ref. [1] have practical importance. After all, in a nuclear system we have additional energy scales, *e.g.* chemical potentials or energies of single-particle excitations, and $m$, or rather the vacuum value of the pion mass, $m_\pi$, need not be small compared to these scales.

In order to gain some quantitative insight we study a model in which the asymmetric nuclear medium is composed of the Fermi gas of constituent quarks of the Nambu–Jona-Lasinio model [3]. Our results are as follows: Firstly, since the model obeys all requirements of current algebra, the chiral soft mode emerges when $m$ is decreased. However, this mode is not necessarily a continuation of the vacuum pion branch. Another branch of low excitations exists ($\sim 10$ MeV), which is analogous to the *spin-isospin sound* [4,5] found in conventional approaches to neutron matter. We show that this mode contributes sizably to current-algebraic sum rules, also at the physical value of $m$.

Let us begin with a reminder of the sum rules of Ref. [1,2]:

$$- 2m\langle \bar{q}q \rangle_C = \sum_{j^-} \mathrm{sgn}(E_{j^-}) \left| \langle j^- | J_{5,0}^-(0)|C\rangle \right|^2 + \sum_{j^+} \mathrm{sgn}(E_{j^+}) \left| \langle j^+| J_{5,0}^+(0)|C\rangle \right|^2, (1)$$

$$2\rho_{I=1} = \sum_{j^-} \frac{1}{|E_{j^-}|} \left| \langle j^- | J_{5,0}^-(0)|C\rangle \right|^2 - \sum_{j^+} \frac{1}{|E_{j^+}|} \left| \langle j^+| J_{5,0}^+(0)|C\rangle \right|^2 , \qquad (2)$$

where $J_{5,0}^a = \bar{q}\gamma_0\gamma_5 \frac{1}{2}\tau^a q$ is the time component of the axial vector current, $|C\rangle$ denotes a uniform medium subject to constraints imposed externally, $| j^\pm \rangle$ are excited states *with three-momentum equal zero* of isospin $\pm 1$, and $E_{j^\pm}$ are their excitation energies. The constraints are the baryon density, $\rho_B$, and the isospin density $\rho_{I=1}$. In Eq. (1) we recognize the generalization of the Gell-Mann–Oakes–Renner sum rule for finite densities. The "isovector" sum rule (2) is nontrivial only for systems with isospin asymmetry. As shown in Ref. [1], since the isovector chemical potential remains finite as $m \to 0$, in a medium with negative isospin density only positive-isospin modes can contribute to the above sum rules in the chiral limit. This implies, under some additional weak assumptions, the existence of a "chiral soft mode" $| + \rangle$ for which

$$E_+ \sim m, \qquad \left| \langle + | J_{5,0}^+ | C \rangle \right| \sim \sqrt{m} \qquad (3)$$

This mode saturates both sum rules in the chiral limit. On the other hand, there are no soft modes with negative isospin, and $E_{j^-} \sim 1$. The presented behavior is radically different from the case of the vacuum or symmetric matter, where the lowest $\pi^+$ and $\pi^-$ excitations scale as $\sqrt{m}$. The $\pi_0$ excitations scale in all cases as $\sqrt{m}$ [1].

Now we pass to an illustration of these general results in a simple model:

the Nambu–Jona-Lasinio model with scalar and vector interactions [3,6]. The model is consistent with current algebra [2], and its variants have been applied extensively to describe both meson and baryon physics. The Lagrangian is

$$\mathcal{L} = \bar{q}(\mathrm{i}\,\partial\!\!\!/ - m)q + \frac{G_\sigma}{2}\left((\bar{q}q)^2 + (\bar{q}\mathrm{i}\gamma_5\tau^a q)^2\right) + \frac{G_\delta}{2}\left((\bar{q}\tau^a q)^2 + (\bar{q}\mathrm{i}\gamma_5 q)^2\right)$$
$$- \frac{G_\rho}{2}\left((\bar{q}\gamma_\mu\tau^a q)^2 + (\bar{q}\gamma_5\gamma_\mu\tau^a q)^2\right) - \frac{G_\omega}{2}(\bar{q}\gamma_\mu q)^2 \ . \tag{4}$$

Using the Hartree approximation one arrives at self-consistency equations for the values of the scalar-isoscalar field $S$, the scalar-isovector field $\delta$, the time component of the neutral vector-isovector field $\rho$, and the time component of the vector-isoscalar field $\omega$ :

$$S = m - G_\sigma\langle\bar{u}u + \bar{d}d\rangle, \ \delta = -G_\delta\langle\bar{u}u - \bar{d}d\rangle,$$
$$\rho = 2G_\rho\langle u^+u - d^+d\rangle, \ \omega = G_\omega\langle u^+u + d^+d\rangle. \tag{5}$$

The scalar and vector densities of the $u$ quark are equal to (analogously for the $d$ quark)

$$\langle\bar{u}u\rangle = 2N_c\int\frac{d^3k}{(2\pi)^3}\frac{M_u}{\sqrt{k^2 + M_u^2}}\Theta_u, \quad \langle u^+u\rangle = 2N_c\int\frac{d^3k}{(2\pi)^3}\Theta(k_u - |k|) \ . \tag{6}$$

We have defined $\Theta_u = \Theta(k_u - |k|) - \Theta(\Lambda - |k|)$, where $\Lambda$ is the sharp three-momentum cut-off used in this paper, and $k_u$ and $k_d$ are the $u$ and $d$ quark Fermi momenta. We have also introduced scalar self-energies $M_u = S + \delta$ and $M_d = S - \delta$. Self-consistency requires that the quark propagators are evaluated with mean-fields (5):

$$S_{u/d}^{-1} = p\!\!\!/ - \gamma_0(\pm\frac{\rho}{2} + \omega) - M_{u/d} + i\varepsilon\,\mathrm{sgn}(\mu_{u/d} - p_0) \ , \tag{7}$$

where $\mu_u$ and $\mu_d$ are the chemical potentials of the $u$ and $d$ quarks. Next, we consider the charged pion propagator. We choose $k_d > k_u$, as for instance in neutron matter. Were there no vector-isovector meson effects (i.e. $G_\rho = 0$), the inverse pion propagator would have a simple form $1 - G_\sigma J_{PP}$, where $J_{PP} = -i\mathrm{Tr}\int\frac{d^4k}{(2\pi)^4}\gamma_5 S_u(p + \frac{1}{2}q)\gamma_5 S_d(p - \frac{1}{2}q)$. However, in presence of vector-isovector coupling there occurs $\pi - A_1$ mixing, and to find excitation energies with pion quantum numbers one has to find *zeros* of the determinant of the inverse $\pi - A_1$ propagator matrix (see e.g. Ref. [2] for details). In the considered model this determinant can be written as

$$D(q_0) = \frac{q_0}{q_0 - \rho}\left\{\frac{m}{S} + \left(\frac{4mSG_\sigma^{-1}G_\rho}{q_0(q_0 - \rho)} - 1\right)\left[G_\sigma J_{PP}(q_0) - \frac{S - m}{S}\right]\right\} \ , \tag{8}$$

where $q_0$ is the frequency of the excitation, and

$$J_{PP}(q_0) = 4N_c \int\limits_{k_u}^{\Lambda} \frac{d^3k}{(2\pi)^3} \frac{(\rho - q_0) + 2\delta M_u/\sqrt{k^2 + M_u^2}}{(\rho - q_0)^2 + 2(\rho - q_0)\sqrt{k^2 + M_u^2} + 4S\delta}$$
$$+ (u \to d, \ \rho \to -\rho, \ \delta \to -\delta, \ q_0 \to -q_0) \ . \tag{9}$$

Let us look at the analytic structure of $D$, or $J_{PP}$, in $q_0$. The state $|C\rangle$ consists of the Fermi seas of $d$ and $u$ quarks, with $k_d > k_u$, as well as of the Dirac sea occupied down to the cut-off $\Lambda$. A positive-charge Fermi sea excitation moves a quark from the occupied $d$ level to an unoccupied $u$ level. Pauli blocking allows this when $\rho + \sqrt{k_d^2 + M_u^2} - \sqrt{k_d^2 + M_d^2} < q_0 < \rho + \sqrt{k_u^2 + M_u^2} - \sqrt{k_u^2 + M_d^2}$. Thus, within these boundaries $D(q_0)$ possesses a cut. There are also additional cuts corresponding to quark-antiquark creation, but their effects are negligible in our study. As explained $e.g.$ in [4,5] in the framework of conventional nuclear physics, it is possible for the pion propagator in neutron matter to have an additional pole at very low excitation energies: the *spin-isospin sound*. We will show that an analogous phenomenon occurs in our model.

For the numerical study we use the following parameter sets:

$$\text{I}: G_\sigma = 7.55\text{GeV}^{-2}, \ G_\delta = 5.41\text{GeV}^{-2}, \ G_\rho = 7.09\text{GeV}^{-2},$$
$$\Lambda = 750\text{MeV}, \ m = 3.52\text{MeV},$$
$$\text{II}: G_\sigma = 4.35\text{GeV}^{-2}, \ G_\delta = 3.34\text{GeV}^{-2}, \ G_\rho = 12.4\text{GeV}^{-2},$$
$$\Lambda = 954\text{MeV}, \ m = 2.03\text{MeV}.$$

Set I is the SU(3) fit of Ref. [7], which fits the $m_\pi = 139\text{MeV}$, $F_\pi = 93\text{MeV}$, $m_\eta = 519\text{MeV}$, and $m_\rho = 765\text{MeV}$. This leaves one parameter out of the original five undetermined, which is expressed through the constituent quark mass in the vacuum, set arbitrarily to $M = 361\text{MeV}$. Set II also fits $m_\pi$, $F_\pi$, $m_\eta$, and $M = 361\text{MeV}$, but not $m_\rho$. Instead, we chose a much larger value of $G_\rho$. Such greater values are needed if, for instance, one wishes to fit the $a_{11}$ pion-pion scattering length [6]. Since the excitations carry no baryon number, the value of $G_\omega$ is irrelevant.

We define $x = \rho_B/\rho_0$, where $\rho_B$ is the baryon density and $\rho_0 = 0.17\text{fm}^{-3}$ is the nuclear saturation density. The relative concentration of $d$ quarks is denoted by $y = \rho_d/(\rho_u + \rho_d)$. For the results presented below we set $y = 2/3$, which corresponds to pure neutron matter. With $y$ fixed, we vary the $x$ variable, increasing the density from 0 to a few times the saturation density. Figure 1 shows the results for the parameter set I. In (a) we plot the mean fields (5). In (b) we show the excitation energies of what we call the "big" modes, the ones that at $x = 0$ (vacuum) become the usual pions. The excitation energy of $\pi^+$ drops to about 100MeV at $x = 2$, and then increases again at

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higher $x$. The excitation energy of $\pi^-$ increases sharply, which is a result of Pauli blocking. At low densities the behavior of these excitation energies is controlled by the Weinberg-Tomosawa term in the $\pi - N$ scattering [2]. The lower part of Fig. 1(b) shows the excitation energy of the *third* solution of the equation $D(q_0) = 0$. This mode is the *spin-isospin sound*, $\pi_s^+$. Its energy is small, rising up to only 6MeV at $x = 2$. The shaded area shows the cut discussed after Eq. (9). The spin-isospin sound emerges from the cut at a very low but finite value of $x$. A similar phenomenon has also been obtained in studies of the kaon propagator in nuclear matter [8,9]. How relevant is this mode? A natural measure of its strength is provided by the contributions to the sum rules (1-2). Explicitly, we have

$$\mathrm{sgn}(E_{j^\pm}) \left| \langle j^\pm | \, J_{5,0}^\pm \, | \, C \rangle \right|^2 = - \left. \frac{2m}{q_0 - \rho} \frac{[SJ_{PP}(q_0) - G_\sigma^{-1}(S - m)]}{dD(q_0)/dq_0} \right|_{q_0 = E_j^\pm} . \ (10)$$

The contributions of the three modes to the sum rules are plotted in Fig. 1(c-d). Whereas in the GMOR sum rule the spin-isospin sound contributes up to 10% (which is still noticeable), in the isovector sum rule it dominates over the "big" modes at $x > 2$, and at $x = 3$ practically saturates the sum rule. In that sense the mode is strong. Note that the three modes satisfy the sum rules at the 99% level, which means that the contributions from cuts are tiny. Next, we present a formal study of the chiral limit. In Fig. 2(a) we plot the excitation energies of the three modes at $x = 2$ and $y = 2/3$ as functions of $\alpha$, which is the ratio of the current quark mass to its physical value from the parameter set I. Hence $\alpha = 1$ corresponds to the physical case, and $\alpha = 0$ is the strict chiral limit. The spin-isospin sound becomes the chiral soft mode of Eq. (3), but only at $\alpha < 10^{-4}$ (!), where its excitation energy scales linearly with $m$. For higher values of $\alpha$ the energy remains flat. The next two plots, Fig. 2(b-c), show the contributions to the sum rules as a function of $\log \alpha$. As required by Eq. (3), the chiral soft mode saturates the sum rules in the chiral limit, and becomes dominant at $\alpha < 10^{-3}$.

Figure 3 shows the results for the parameter set II. The important qualitative difference when compared to Fig. 1 may be seen in the bottom of Fig. (c): the excitation energy of the spin-isospin sound is negative now. As discussed in Ref. [1], the energy of a charged excitation in asymmetric medium need not be positive. This is because we require that the state $|C\rangle$ be the ground state for a given value of the constraint, here the isospin. A charged excitation connects $|C\rangle$ to states with different isospin, hence such an excitation may in fact be a deexcitations. More explicitly, we note that the $u$ quark is lighter than $d$, because the difference of their vector self energies, $\rho$, which is negative, overcomes the positive difference of the scalar self energies (cf. Eq (7)). Therefore changing $u$ to $d$ is favorable energetically. Note that this negative excitation is still above the chemical potential for isospin, which is equal to

$\mu_u - \mu_d = \rho + \sqrt{k_u^2 + M_u^2} - \sqrt{k_d^2 + M_d^2}$, hence it carries *positive* isospin. Thus the system may lower its energy by spontaneously emitting the $\pi_s^+$ mode. However, by charge conservation this has to be accompanied by the emission of a particle of negative charge. The strong-interaction modes ($\pi^-$, or $\bar{u}d$ pair) are heavy enough, so the system remains stable with respect to strong decays. In weak decays the accompanying particle is the electron or muon. Depending on the value of the chemical potential of these particles, the reaction may lead to a net energy gain. In that case the system is unstable with respect to weak interactions.

Figures 3(c-d) show the sum rules for the parameter set II. Again, we notice that the spin-isospin sound mode contributes at the level of 10-20% to the GMOR sum rule, but with opposite sign. It dominates the isovector sum rule at $x > 3$. Figure 4 shows a qualitatively different behavior of the system close to the chiral limit for the parameter set II than for the set I. We see that now it is the "big" $\pi^+$ mode, not the $\pi_s^+$, which assumes the role of the chiral soft mode. The $\pi_s^+$ has a finite energy in the chiral limit, and does not contribute to the sum rules. These are dominated by the "big" $\pi^+$ mode at $\alpha < 10^{-3}$, where it scales linearly with $m$.

The linear dependence of the excitation energy of the chiral soft mode on the current quark mass takes effect only at extremely small values of $m$, corresponding to $m_\pi$ 10—100 times lighter than the physical value. Hence the satisfaction of the laws (3) has formal rather than practical meaning. Nevertheless, it is interesting that these laws may be fulfilled by the highly collective spin-isospin sound, rather than by the vacuum pion branch. In our model at all values of $m$ and densities the three pionic modes saturate the current-algebra sum rules at the 99% level, and the contributions from the spin-isopin sound are big. This shows that in constructing an effective theory of mesons in medium one should account for such excitations, including possible colective modes, and that the contribution from particle-hole excitations are negligible. It is challenging to see if similar results hold in more realistic approaches to nuclear matter, with nucleon degrees of freedom. In particular, the role of short-range correlations should be examined, since they are known affect strongly the excitation spectra in nuclear matter.

## References

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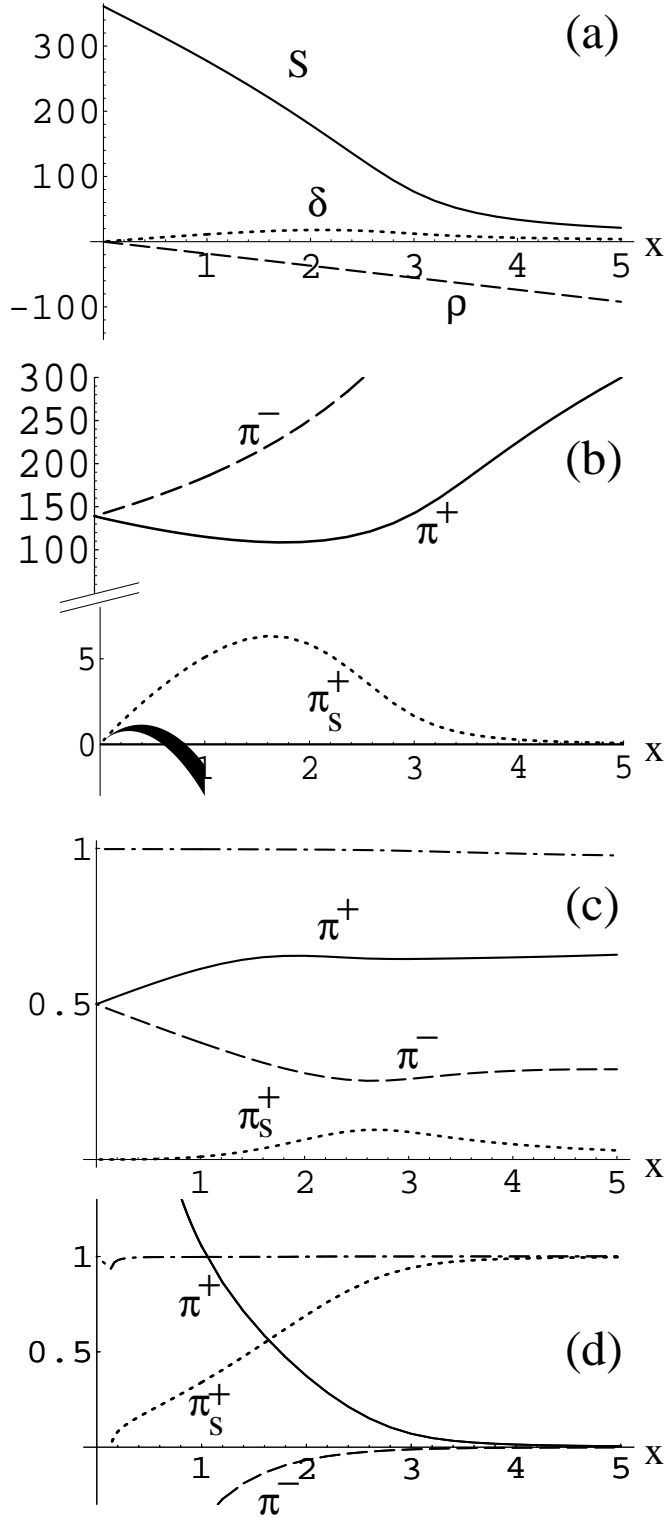[9] H. Yabu, S. Nakamura, F. Myhrer, and K. Kubodera, Phys. Lett. **B 315** (1993) 17.

Fig. 1. Dependence of various quantities on $x = \rho_B/\rho_0$ for the parameter set I and $y = 2/3$: (a) Mean fields in MeV. (b) Excitation energies for the "big" modes (top) and the spin-isospin sound (bottom) in MeV. Also shown is the particle-hole production cut (shaded region). (c) The relative contributions of the three modes to the GMOR sum rule. The sum (dash-dot line) is indistinguishable from one. (d) Same as (c) for the isovector sum rule.
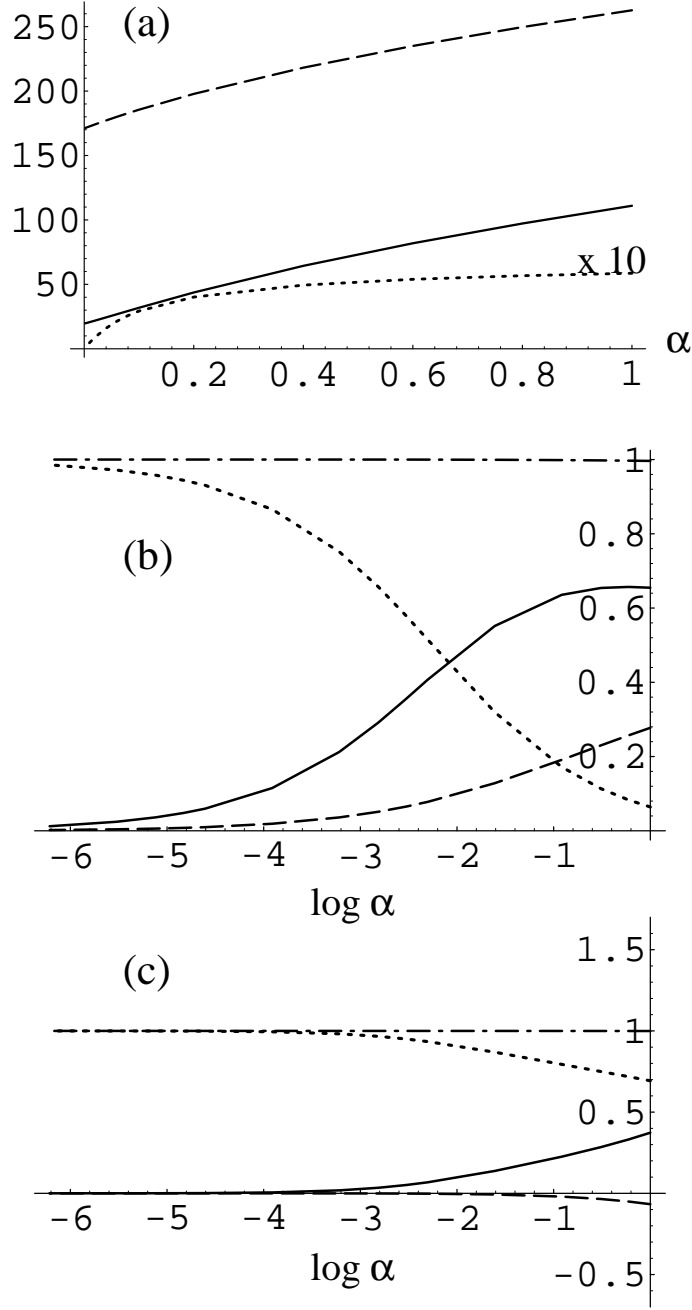
Fig. 2. Study of the chiral limit. Convention for lines as in Fig. 1(b). (a) Dependence of excitation energies of the three modes on the ratio of the current quark mass to its physical value, $\alpha$. The curve for the spin-isospin sound has been multiplied by 10. (b) Relative contributions of the three modes to the GMOR sum rule. (c) Relative contributions of the three modes to the isovector sum rule. The spin-isospin sound becomes the chiral soft mode.
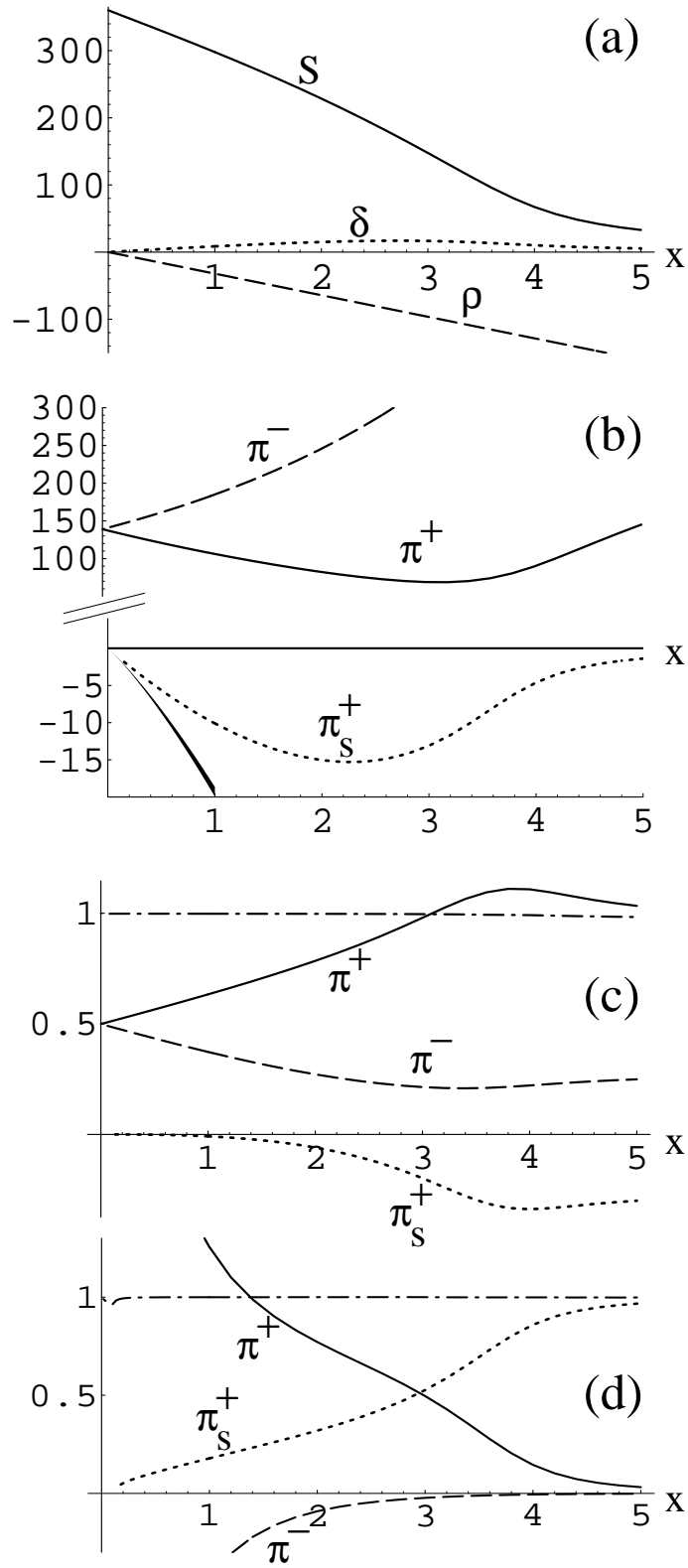
Fig. 3. Same as Fig. 1 for the parameter set II. The spin-isospin sound mode in (b) has negative excitation energy.
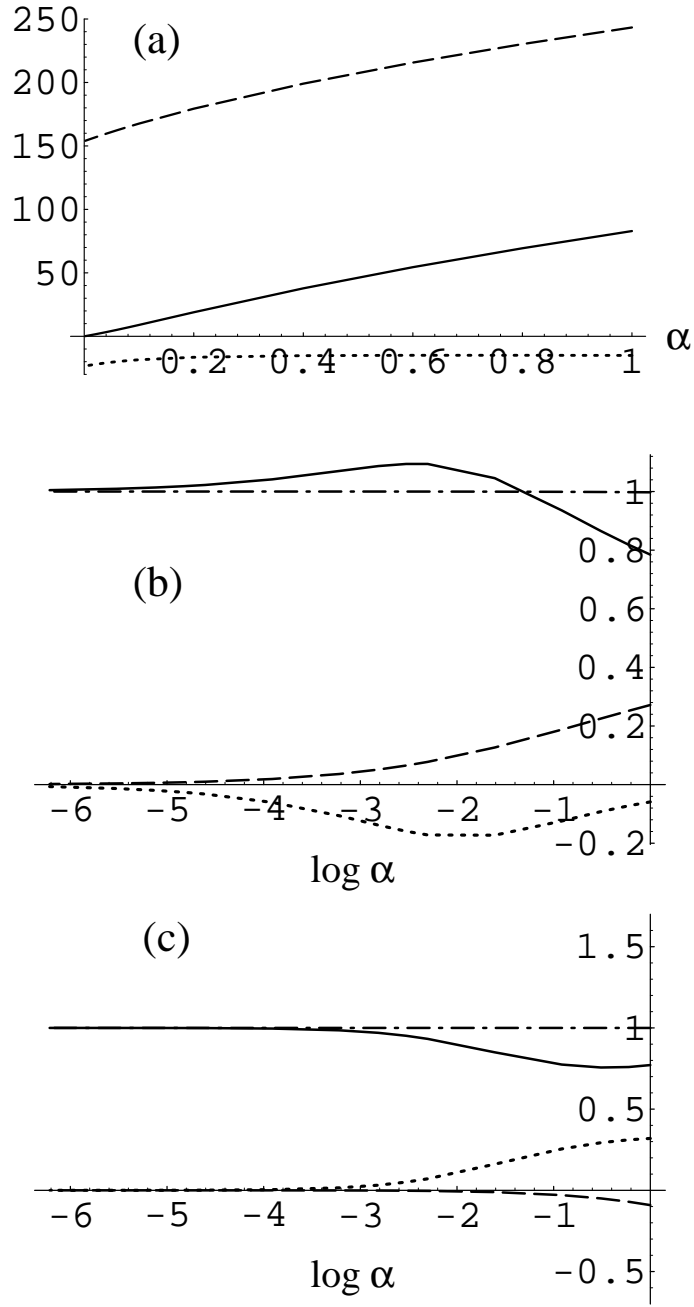
Fig. 4. Same as Fig. 2 for the parameter set II. The "big" mode becomes the chiral soft mode.